An intuitive explanation for Proposition 3.1

In our 2002 paper "Separating equilibria in continuous signalling games", we present a general method for finding the separating equilibria of a continuous signalling game as integral curves of a vector field determined by the game's pay-off structure. At the heart of this method is the result presented in Proposition 3.1 of that paper:

Proposition 1 *Define the vector field V as*

$$V(s', q') = \left(\frac{d}{ds} C(q', s) \bigg|_{s=s'} \; , \; \frac{d}{dq} H(q', R_q^*(q)) \bigg|_{q=q'} \right).$$

If a separating equilibrium exists for the game Γ with $S(q_0) = s_0$, the integral curve of V through (q_0, s_0) will be an equilibrium signalling strategy $S(q)$, provided that everywhere along this integral curve...[a second-order condition given in the paper]...is satisfied. The equilibrium receiver strategy is given by $R(s) = R_q^* \left(S^{-1} (q) \right)$ where $S^{-1}(q)$ is the inverse of S.

(As a terminological refresher, q is quality, s is signal intensity, r is response. $C(q, s)$ is the signal cost function, $H(q, r)$ is the benefit function, $s = S(q)$ is a signalling strategy and $r = R(s)$ is a response strategy.)

In the paper we provide a rather complicated proof of this proposition. A shorter intuitive explanation would be useful. Here we provide such an explanation.

For the signalling strategy $s = S(q)$ to be an equilibrium, the marginal cost to the signaller of changing a signal must be equal to the marginal benefit everywhere along this equilibrium strategy path.

$$\frac{\partial C}{\partial s} = \frac{\partial H}{\partial r} \frac{dR}{ds} \tag{1}$$

We aim to find a vector field such that an equilibrium signalling strategy $S(q)$ is an integral curve of that vector field. That is, we want to know how the signal changes as the signaller's quality changes: dS/dq. Let's multiply both sides of (1) by this:

$$\frac{\partial C}{\partial s} \frac{dS}{dq} = \frac{\partial H}{\partial r} \frac{dR}{ds} \frac{dS}{dq} \tag{2}$$

Note that the right hand side above is the marginal benefit to a change in the signaller's *perceived* quality, because we are only looking at how change in quality affects change in benefit through change in signal.

Although the value of dR/ds by itself is unknown and the value of dS/dq is what we are looking for, we do know the product of these two partials, because we know that at equilibrium the composition of R and S yields the optimal response to each level of quality: $R(S(q)) = R^*(q)$. Thus we can write:

$$\frac{\partial C}{\partial s}\frac{dS}{dq} = \frac{\partial H}{\partial r}\frac{dR^*}{dq} \tag{3}$$

Finally, we can solve for dS/dq as desired:

$$\frac{dS}{dq} = \frac{\partial H}{\partial r}\frac{dR^*}{dq}/\frac{\partial C}{\partial s} \tag{4}$$

Then S(q) will be an integral curve of the vector field given by $(\frac{\partial C}{\partial s}, \frac{\partial H}{\partial r}\frac{dR^*}{dq})$. These terms are equal to the total derivatives given in Proposition 3.1.

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